(A): Chapter 1;

(A-B): Chapter 2, Chapter 4, appendix;

(B): Chapter 3, Chapter 5.

The presentation of the material on infinite series was inspired by notes used at the Technical University of Denmark. In their original version, these notes were written by H. E. Jensen; several professors from the Department of Mathematics have contributed to the later versions. Some of our examples are borrowed from these notes.

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In the present reprint, a small number of misprints are corrected. Example 3.2.6 is modified, and Exercises 2.13, 2.14, 2.15 and 3.10 are new.

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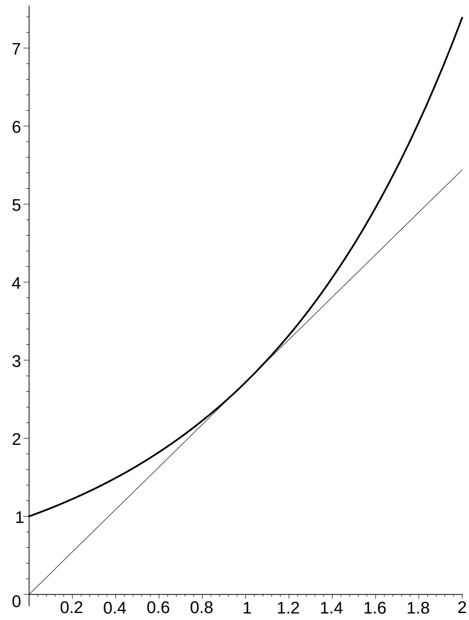

Figure 1.3.4 *The function $f(x) = e^x$ together with the first Taylor polynomial at $x_0 = 1$ (thin line).*

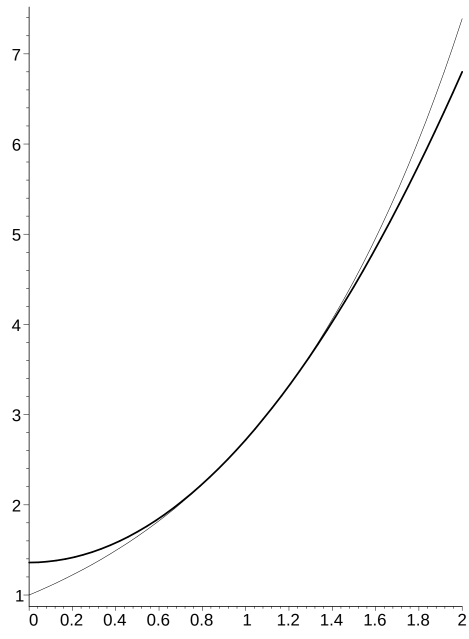

Figure 1.3.5 *The function $f(x) = e^x$ (thin curve) together with the second Taylor polynomial at $x_0 = 1$.*

We now give a geometric argument showing that

$$\sum_{n=1}^{N} \frac{1}{2^n} = 1 - \frac{1}{2^N}. \tag{2.4}$$

In fact, $S_1 = \frac{1}{2}$, which is exactly half of the distance from 0 to 1. That is,

$$S_1 = 1 - \frac{1}{2}.$$

Further, $S_2 = \frac{1}{2} + \frac{1}{2^2}$; thus, the extra contribution compared to S_1 is $\frac{1}{2^2}$, which is half the distance from S_1 to 1. Going to the next step, i.e., looking at S_3, again cuts the distance to 1 by a factor of 2:

$$S_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1 - \frac{1}{2^3}.$$

The same happens in all the following steps: when looking at S_N, i.e., after N steps, the distance to 1 will be exactly $1/2^N$, which shows (2.4). It follows that

$$\sum_{n=1}^{N} \frac{1}{2^n} \to 1 \text{ for } N \to \infty;$$

we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is convergent with sum 1.}$$

Concerning the series in (iii), we first observe that for any $n \in \mathbb{N}$,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1};$$

thus, the Nth partial sum is

$$\begin{aligned} S_N &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{N(N+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

Since $S_N \to 1$ as $N \to \infty$, we conclude that the series is convergent with sum 1. □

Intuitively, the "problem" with the series in Example 2.1.2 (i) is that the terms $a_n = n$ are too large: considering partial sums S_N for larger and larger values of N means that we continue to add numbers to S_N, and when they grow as in this example, it prevents S_N from having a finite limit as $N \to \infty$. A refinement of this intuitive statement leads to a useful criterion, Since

$$\frac{r^{N+1}}{1-r} \to 0 \text{ as } N \to \infty,$$

we can thus make $\left|f(x) - \sum_{n=0}^{N} x^n\right|$ as small as we want, simultaneously for all $x \in [-r, r]$, by choosing $N \in \mathbb{N}$ sufficiently large. $\square$

Example 2.6.1 suggests that we introduce the following type of convergence.

Definition 2.6.2 *Given functions $f_1, f_2, \ldots, f_n, \ldots$ defined on an interval I, assume that the function*

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \ \ x \in I,$$

is well defined. Then we say that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on the interval I if for each $\epsilon > 0$ we can find an $N_0 \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \le \epsilon \text{ for all } \ x \in I \text{ and all } N \ge N_0.$$

Note that an equivalent definition would be to say that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to f on the interval I if for each $\epsilon > 0$ we can find $N_0 \in \mathbb{N}$ such that

$$\sup_{x \in I} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \le \epsilon \text{ for all } N \ge N_0.$$

Let us return to Example 2.6.1:

Example 2.6.3 Formulated in terms of uniform convergence, our result in Example 2.6.1 shows that

- $\sum_{n=0}^{\infty} x^n$ does not converge uniformly to $f(x) = \frac{1}{1-x}$ on $I =]-1, 1[$;
- $\sum_{n=0}^{\infty} x^n$ converges uniformly to $f(x) = \frac{1}{1-x}$ on any interval $I = [-r, r], \ r \in]0, 1[$.

It turns out that Example 2.6.3 is typical for power series: in general, nothing guarantees that a power series with radius of convergence ρ converges uniformly on $]-\rho, \rho[$. On the other hand we always obtain uniform convergence if we shrink the interval slightly:

Proposition 2.6.4 *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $\rho > 0$. Then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on any interval of the form $[-r, r]$, where $r \in]0, \rho[$.*

Proof: Given any $x \in [-r, r]$,

$$\begin{aligned} \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n \right| &= \left| \sum_{n=N+1}^{\infty} a_n x^n \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n x^n| \\ &\leq \sum_{n=N+1}^{\infty} |a_n r^n|\,; \end{aligned}$$

thus

$$\sup_{x\in[-r,r]} \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n \right| \leq \sum_{n=N+1}^{\infty} |a_n r^n|\,. \tag{2.26}$$

By the fact that $\sum_{n=1}^{\infty} a_n r^n$ is absolutely convergent, the quantity on the right-hand side of (2.26) goes to zero as $N \to \infty$. □

We end this section with a few important results concerning continuity and differentiability of infinite series of functions; the first result below gives conditions for the associated sum function being well defined and continuous.

Theorem 2.6.5 *Assume that the functions $f_1, f_2, \ldots$ are defined and continuous on an interval I, and that there exist positive constants $k_1, k_2, \ldots$ such that*

(i) $|f_n(x)| \leq k_n, \ \forall x \in I, \ n \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} k_n$ is convergent.

Then the infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly; and the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \ \ x \in I$$

is continuous.

A series $\sum_{n=1}^{\infty} k_n$ satisfying the condition (i) in Theorem 2.6.5 is called a *majorant series* for $\sum_{n=1}^{\infty} f_n(x)$. Theorem 2.6.5 is known in the literature under the name *Weierstrass' M-test.*

If we want the sum function to be differentiable, we need slightly different conditions, stated below; on the theoretical level, the result can be used to prove Theorem 2.4.10.

Theorem 2.6.6 *Assume that the functions $f_1, f_2, \ldots$ are defined and differentiable with a continuous derivative on the interval I, and that the* *function* $f(x) = \sum_{n=1}^{\infty} f_n(x)$ *is well defined on* I*. Assume also that there exist positive constants* $k_1, k_2, \dots$ *such that*

(i) $|f_n'(x)| \le k_n, \ \forall x \in I, \ n \in \mathbb{N}$*;*

(ii) $\sum_{n=1}^{\infty} k_n$ *is convergent.*

Then f *is differentiable on* I*, and*

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Finally, we mention that uniform convergence of an infinite series consisting of integrable functions allows us to integrate term-wise:

Proposition 2.6.7 *Assume that the functions* $f_1, f_2, \dots$ *are continuous on the interval* I *and that* $\sum_{n=1}^{\infty} f_n(x)$ *is uniformly convergent. Then, the function* $x \mapsto \sum_{n=1}^{\infty} f_n(x)$ *is continuous on* I*; furthermore, for any* $a, b \in I$*,*

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

Proposition 2.6.7 has an important consequence for integration of power series:

Corollary 2.6.8 *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $\rho > 0$*. Then, for any* $b \in]-\rho, \rho[$*,*

$$\int_0^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} b^{n+1}.$$

2.7 Signal transmission

Modern technology often requires that information can be sent from one place to another; one speaks about *signal transmission.* It occurs, e.g., in connection with wireless communication, the internet, computer graphics, or transfer of data from CD-ROM to computer.

All types of signal transmission are based on transmission of a series of numbers. The first step is to convert the given information (called the *signal*) to a series of numbers, and this is where the question of having a series representation comes in: if we know that a signal is given as a function f which has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{2.27}$$

2.12 Find the values of $x \in \mathbb{R}$ for which

$$\sum_{n=0}^{\infty} \frac{1}{(2+x^2)^n}$$

is convergent, and express the sum via the standard functions.

2.13 Prove that

$$\sum_{n=k+1}^{\infty} \frac{1}{(2n-1)^2} \leq \frac{1}{4k+2} + \frac{1}{(2k+1)^2}, \ \forall k \in \mathbb{N}.$$

2.14 Find a polynomial $P(x)$ such that

$$|\sin x - P(x)| \leq 0.05 \text{ for all } \ x \in [0,3].$$

2.15 Assume that we want to transmit a function having a power series expansion. Explain why uniform convergence of the power series is relevant in this context.

The importance of Theorem 3.2.3 lies in the fact that it shows how a large class of functions can be decomposed into a sum of elementary sine and cosine functions. As a more curious consequence we mention that this frequently allows us to determine the exact sum of certain infinite series:

Example 3.2.4 For the step function in (3.4), Example 3.1.2 implies that

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x, \ \forall x \in \mathbb{R}. \tag{3.9}$$

For $x \notin \pi\mathbb{Z} := \{0, \pm\pi, \pm 2\pi, \dots\}$, this result follows from Theorem 3.2.3 (i). For $x \in \pi\mathbb{Z}$ it follows from (ii) in the same theorem and the special definition of $f(0)$: a different choice of $f(0)$ would not change the Fourier series, but (3.9) would no longer hold for $x \in \pi\mathbb{Z}$.

Applying (3.9) with $x = \pi/2$ shows that

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)\frac{\pi}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} (-1)^{n-1},$$

or,

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} (-1)^{n-1} = \frac{\pi}{4}. \qquad \square$$

For a continuous function f, the assumptions in Theorem 3.2.3 imply that the Fourier series converges uniformly to f. Equation (3.8) can be used to estimate how many terms we need to keep in the Fourier series in order to guarantee a certain approximation of the function f: if we want that $|f(x) - S_N(x)| \leq \epsilon$ for a certain $\epsilon > 0$, we can choose N such that

$$\frac{1}{\sqrt{N}} \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\pi}^{\pi} |f'(t)|^2 dt} \leq \epsilon,$$

i.e.,

$$N \geq \frac{\int_{-\pi}^{\pi} |f'(t)|^2 dt}{\pi \epsilon^2}. \tag{3.10}$$

Note that (3.10) is a "worst case estimate": it gives a value of $N \in \mathbb{N}$ which can be used for all functions satisfying the conditions in Theorem 3.2.3. In order to minimize the calculation cost we usually want to obtain a given approximation using as small values of N as possible. In concrete cases where the Fourier coefficients are known explicitly, the next result can often be used to prove that smaller values of N than suggested in (3.10) are sufficient.

Proposition 3.2.5 *Assume that f is continuous, piecewise differentiable and 2π-periodic, with Fourier coefficients a_n, b_n. Then*

$$|f(x) - S_N(x)| \le \sum_{n=N+1}^{\infty} (|a_n| + |b_n|), \ \forall x \in \mathbb{R}. \tag{3.11}$$

Proof: By Theorem 3.2.3, the assumptions imply that the Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$. Via (3.7) and (3.3),

$$\begin{aligned} |f(x) - S_N(x)| &= \Bigg| \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &\quad - \left(\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \right) \Bigg| \\ &= \left| \sum_{n=N+1}^{\infty} (a_n \cos nx + b_n \sin nx) \right| \\ &\le \sum_{n=N+1}^{\infty} |a_n \cos nx + b_n \sin nx| \\ &\le \sum_{n=N+1}^{\infty} (|a_n| + |b_n|). \end{aligned}$$

□

In the next example we compare Theorem 3.2.3 and Proposition 3.2.5.

Example 3.2.6 Consider the 2π-periodic function given by

$$f(x) = |x|, \ x \in [-\pi, \pi[.$$

Our purpose is to find estimates for $N \in \mathbb{N}$ such that

$$|f(x) - S_N(x)| \le 0.1 \text{ for all } x \in \mathbb{R}. \tag{3.12}$$

The reader can check (Exercise 3.10) that the Fourier series of f is given by

$$f \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x.$$

See Figures 3.2.7–3.2.8, which show the function f and some partial sums. According to Theorem 3.2.3, the Fourier series converges uniformly to f.

We first apply (3.10), which was derived as a consequence of Theorem 3.2.3: it shows that (3.12) is satisfied if

$$N \ge \frac{2\pi}{0.1^2\pi} = 200.$$

Let us now apply Proposition 3.2.5. First, for any $k \in \mathbb{N}$, the partial sum

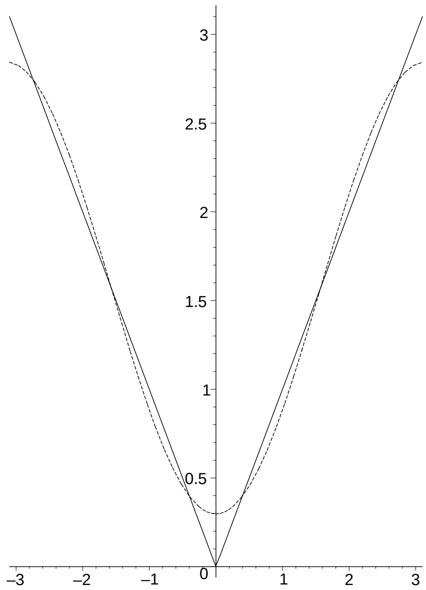

Figure 3.2.7 *The function $f(x)) = |x|$ and the partial sum $S_3(x)$, shown on the interval $[-\pi, \pi]$.*

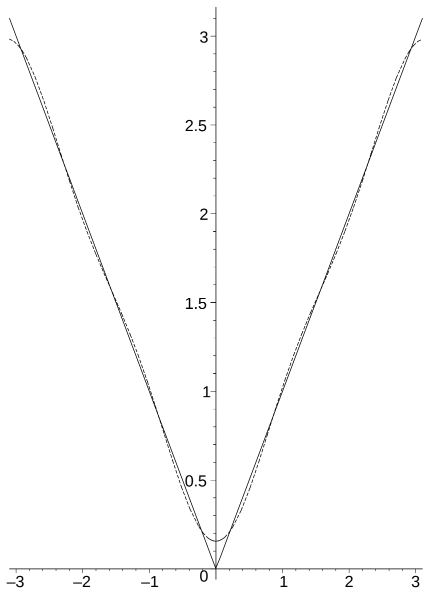

Figure 3.2.8 *The function $f(x)) = |x|$ and the partial sum $S_5(x)$, shown on the interval $[-\pi, \pi]$.*

$S_{2k-1}(x)$ of the Fourier series is given by

$$S_{2k-1}(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{k} \frac{1}{(2n-1)^2}\cos(2n-1)x.$$

Note also that $S_{2k-1}(x) = S_{2k}(x)$. Via (3.11),

$$|f(x) - S_{2k-1}(x)| = \left|\frac{4}{\pi}\sum_{n=k+1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x\right| \leq \frac{4}{\pi}\sum_{n=k+1}^{\infty}\frac{1}{(2n-1)^2}.$$

Applying the result in Exercise 2.13, we obtain that this is satisfied if

$$\frac{4}{\pi}\left(\frac{1}{4k+2} + \frac{1}{(2k+1)^2}\right) \leq 0.1,$$

i.e., for $k \geq 4$. Thus, (3.12) is satisfied for the partial sum $S_N(x)$ as soon as $N \geq 2k - 1 = 7$.

For the function considered here we see that Proposition 3.2.5 leads to a much better result than Theorem 3.2.3. The difference between the estimates obtained via these two results is getting larger when we ask for better precision: if we decrease the error-tolerance by a factor of 10,

- Theorem 3.2.3 will increase the value of N by a factor of 100;
- Proposition 3.2.5 will increase the value of N by a factor of approximately 10.

See Exercise 3.10. We note that this result is based on the choice of the considered function: the difference between the use of Theorem 3.2.3 or Proposition 3.2.5 depends on the given function. □

3.3 Fourier series and signal analysis

Fourier's theorem tells us that the Fourier series for a continuous piecewise differentiable and 2π-periodic function converges pointwise toward the function; that is, we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx),\ x \in \mathbb{R}. \tag{3.13}$$

Our aim is now to describe how to interpret this identity.

If we think about the variable x as time, the terms in the Fourier series correspond to oscillations with varying frequencies: for a given value of $n \in \mathbb{N}$, the functions $\cos nx$ and $\sin nx$ run through $\frac{n}{2\pi}$ periods in a time interval of unit length, i.e., they correspond to oscillations with frequency $\nu = \frac{n}{2\pi}$. Now, given a function f, the identity (3.13) represents f as a superposition involving a potential limit. In general infinite-dimensional normed vector spaces, the situation is different: there might exist Cauchy sequences which are not convergent. Hilbert spaces are introduced in order to avoid this type of complication:

Definition 3.4.1 *A Hilbert space is a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, with the property that each Cauchy sequence is convergent.*

Every finite-dimensional vector space with an inner product is a Hilbert space; and one can view Hilbert spaces as an extension of this framework, where we now allow certain infinite-dimensional vector spaces. In Hilbert spaces, several of the main results from linear algebra still hold; however, in many cases, extra care is needed because of the infinitely many dimensions. The exact meaning of this statement will be clear soon.

In Hilbert spaces, one can also define orthonormal bases, which now become infinite sequences $\{e_k\}_{k=1}^{\infty}$; one characterization is that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ if

$$||f||^2 = \sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2, \ \forall f \in \mathcal{H} \ \text{ and } \ \langle e_k, e_j \rangle = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases} \tag{3.16}$$

If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then one can prove that each $f \in \mathcal{H}$ has the representation

$$f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k. \tag{3.17}$$

At a first glance, this representation looks similar to (3.14), but some words of explanation are needed: in fact, the vectors $\{e_k\}_{k=1}^{\infty}$ belong to an abstract vector space, so we have not yet defined what an infinite sum like (3.17) should mean! The exact meaning appears by a slight modification of our definition of convergence for a series of numbers, namely that we measure the difference between f and the partial sums of the series in (3.17) in the norm $|| \cdot ||$ associated to our Hilbert space. That is, (3.17) means by definition that

$$\left\| f - \sum_{k=1}^{N} \langle f, e_k \rangle e_k \right\| \to 0 \text{ as } N \to \infty.$$

In our context, the important fact is that (if we identify functions which are equal almost everywhere) $L^2(-\pi, \pi)$ is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx, \ f, g \in L^2(-\pi, \pi). \tag{3.18}$$

A very important fact about the Fourier transform is its invertibility: if we happen to know the Fourier transform $\hat{f}$ of an, in principle, unknown function f, we are able to come back to the function f via

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i x \gamma} d\gamma. \tag{3.37}$$

An exact statement of this result will need some more care; for us, it is enough to know that this formula holds at least if we know that f is a continuous function in $L^1(\mathbb{R})$ which vanishes at $\pm\infty$.

3.10 Exercises

3.1 Consider the 2π-perodic function f given by

$$f(x) = \frac{1}{4}x^2 - \frac{\pi}{2}x, \;\; x \in [0, 2\pi].$$

(i) Prove that f is an even function.

(ii) Find the Fourier coefficients.

(iii) Find the sums of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(iv) Find for $x \in [0, 2\pi]$ the sum of the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

3.2 Let f be the 2π-periodic function given by

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{\pi}{4}[, \\ 0 & \text{for } x \in [\frac{\pi}{4}, 2\pi[. \end{cases}$$

(i) Find the Fourier series for f.

(ii) Does the Fourier series have a convergent majorant series?

(iii) How can you answer (ii) without looking at the Fourier series?

3.3 Consider the 2π-periodic function f given by

$$f(x) = \begin{cases} \sin x & \text{if } \;\; 0 < x \leq \pi, \\ 0 & \text{if } \;\; \pi < x \leq 2\pi. \end{cases}$$

3.8 (i) Prove that

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n-1)(2n+1)},\ \forall x\in\mathbb{R}.$$

(ii) Calculate the number $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$.

(iii) Calculate the number $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2(2n+1)^2}$.

(iv) Write the Fourier series for $|\sin(\cdot)|$ in complex form.

(v) Compare the decay of the coefficients in the Fourier series for $\sin(\cdot)$ and $|\sin(\cdot)|$; see Theorem 3.7.2.

(vi) Denote the Nth partial sum of the Fourier series for $|\sin(\cdot)|$ by S_N. Find N such that

$$|\,|\sin x| - S_N(x)| \le 0.1,\ \forall x\in\mathbb{R}.$$

3.9 Consider the odd 2π-periodic function, which for $x\in[0,\pi]$ is given by

$$f(x) = \frac{\pi}{96}(x^4 - 2\pi x^3 + \pi^3 x).$$

(i) Find $f(-\frac{\pi}{2})$.

(ii) Prove that

$$f(x) = \sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^5},\ x\in\mathbb{R}$$

(hint: $\int_0^{\pi}(x^4 - 2\pi x^3 + \pi^3 x)\sin nx\,dx = 24\frac{1-(-1)^n}{n^5},\ n\in\mathbb{N}$).

(iii) Prove that

$$|f(x) - \sin x| \le 0.01,\ \forall x\in\mathbb{R}$$

(hint: use the integral test).

3.10 This exercise supplements Example 3.2.6.

(i) Prove that the Fourier series in Example 3.2.6 has the announced form, and that the sign "$\sim$" can be replaced by "$=$".

(ii) Argue for the expression for $S_{2k-1}(x)$ and for the fact that $S_{2k-1}(x) = S_{2k}(x)$.

(iii) Use the two methods in Example 3.2.6 to find estimates for $N\in\mathbb{N}$ such that $|f(x) - S_N(x)| \le 0.01$ for all $x\in\mathbb{R}$.

$$\begin{aligned}
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \ |x| < 1 \\
\sinh x &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \ x \in \mathbb{R} \\
\cosh x &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \ x \in \mathbb{R}
\end{aligned}$$

B.2 Fourier series for 2π-periodic functions

$$f(x) = \begin{cases} -1 \text{ if } x \in [-\pi, 0[, \\ 0 \text{ if } x = 0, \\ 1 \text{ if } x \in]0, \pi[, \end{cases} \quad , \quad f \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x \tag{3.5}$$

$$f(x) = x, \ x \in]-\pi, \pi[: \quad f \sim \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx \tag{3.6}$$

$$f(x) = |x|, \ x \in]-\pi, \pi[: \ f \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x$$

$$f(x) = x^2, \ x \in]-\pi, \pi[: \quad f \sim \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{n+1} \cos nx$$

$$f(x) = |\sin x|, \ x \in]-\pi, \pi[: \quad f \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)}$$

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